Duality results in the homogenization of two-dimensional high-contrast conductivities

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Abstract

The paper deals with some extensions of the Keller-Dykhne duality relations arising in the classical homogenization of two-dimensional uniformly bounded conductivities, to the case of high-contrast conductivities. Only assuming a L^1-bound on the conductivity we prove that the conductivity and its dual converge respectively, in a suitable sense, to the homogenized conductivity and its dual. In the periodic case a similar duality result is obtained under a less restrictive assumption.

1 Introduction

The homogenization of elliptic partial differential equations has had an important development for nearly forty years. During the seventies, the G-convergence of Spagnolo [24], and the H-convergence of Murat, Tartar [25], [23], as well as the study of periodic structures by Bensoussan, Lions, Papanicolaou [4] (see also [15]), laid the foundations of the homogenization theory in conduction problems with uniformly bounded (both from below and above) conductivities.

The boundedness assumption implies some compactness which preserves the nature of the homogenized problem. This is no more the case for high-contrast conductivities. Indeed, Khruslov was one of the first to derive vector-valued homogenized problems in the case of low conductivities [17], as well as nonlocal homogenized ones in the case of high conductivities [12] (see also [18] and [19] for various types of homogenized problems and complete references). In the case of high conductivities, the appearance of nonlocal effects is strongly linked to the dimension greater than two. So, the model example of nonlocal homogenization [12] in conduction is obtained from a three-dimensional homogeneous medium reinforced by highly conducting thin fibers which create a capacitary effect (see also [3], [6] and [10] for extensions and alternative methods).

Recently, Casado-Díaz and the first author proved in [5], [8], [9], that dimension two, contrary to dimension three or greater, induces an extra compactness which prevents from the nonlocal effects. In particular, an extension of the H-convergence is obtained in [8] for conductivities which are only bounded in L^1 but not in L^∞.

The present paper deals with the duality relations arising in the two-dimensional homogenization. These relations were first noted by Keller [16] who obtained an interchange equality relating the effective properties of a two-phase composite when the conductivities are swapped. Following the pioneer work of Keller, Dykhne [11] (see also [21] and [13] for a more general approach) proved that, for any periodic, coercive and bounded matrix-valued function A, the homogenized matrix associated with the dual conductivity $A^T/\det A$ (where A^T denotes the transposed of A) is equal to $A_*^T/\det A_*$, where A_* is the constant homogenized matrix associated with A. We refer to Chapters $3, 4$ of [22] for a general presentation of the duality transformations.

Our contribution is the extension of the Dykhne duality relation to high-contrast two-dimensional conductivities. More precisely, consider an equicoercive sequence A_n of (not necessarily symmetric) conductivity matrices, which is not uniformly bounded contrary to the classical case. Under the main assumption that

$$\frac{\det A_n}{\det A_n^s}\,|A_n^s| \text{ weakly-}* \text{ converges in the sense of the Radon measures to a bounded function,} \tag{1.1}$$

(where A_n^s denotes the symmetrized of A_n), we prove (see Theorem 2.2) that the sequence $A_n^T/\det A_n$ "H-converges" to $A_*^T/\det A_*$, when A_n "H-converges" to A_*, for suitable extensions of the H-convergence (see Definition 2.1). As a consequence, we obtain (see Corollary 2.4) a compactness result for the opposite case of a uniformly bounded but not equicoercive sequence of conductivity matrices. We also prove a refinement (see Theorem 2.7) in the periodic case, i.e. $A_n(x) := A_n^\sharp(\frac{x}{\varepsilon_n})$ where $A_n^\sharp$ is Y-periodic and $\varepsilon_n > 0$ tends to 0, under the less restrictive assumption than (1.1)

$$\varepsilon_n^2 \int_Y \frac{\det A_n^\sharp}{\det(A_n^\sharp)^s}\left|(A_n^\sharp)^s\right| dy \underset{n\to+\infty}{\longrightarrow} 0. \tag{1.2}$$

The paper is organized as follows. In Section 2, we define some appropriate notions of H-convergence and we state the main duality results for high-contrast conductivities, both in the non-periodic and periodic framework. Section 3 is devoted to the proof of the homogenization results.

Notations

- Ω denotes a bounded open subset of $\mathbb{R}^2$;
- I denotes the unit matrix in $\mathbb{R}^{2\times 2}$, and J the rotation matrix of angle $90°$;
- for any matrix A in $\mathbb{R}^{2\times 2}$, A^T denotes the transposed of the matrix A, A^s denotes its symmetric part in such a way that $A = A^s + aJ$, where $a \in \mathbb{R}$;
- for any matrices $A, B \in \mathbb{R}^{2\times 2}$ (even non-symmetric), $A \le B$ means that $A^s \le B^s$, i.e., for any $\xi \in \mathbb{R}^2$, $A\xi\cdot\xi \le B\xi\cdot\xi$;
- $|\cdot|$ denotes both the euclidian norm in $\mathbb{R}^d$ and the subordinate norm in $\mathbb{R}^{2\times 2}$, i.e., for any $A \in \mathbb{R}^{2\times 2}$, $|A| := \sup\{|Ax| : |x| = 1\}$, which agrees with the spectral radius of A if A is symmetric;
- for any $\alpha, \beta > 0$, $M(\alpha,\beta;\Omega)$ denotes the set of the matrix-valued functions $A : \Omega \longrightarrow \mathbb{R}^{2\times 2}$ such that

$$\forall\,\xi \in \mathbb{R}^2, \quad A(x)\xi\cdot\xi \ge \alpha\,|\xi|^2 \quad \text{and} \quad A^{-1}(x)\xi\cdot\xi \ge \beta^{-1}\,|\xi|^2, \quad \text{a.e. } x \in \Omega; \tag{1.3}$$

- for $Y := (0,1)^2$ and for $V := L^p, W^{1,p}$, $V_\#(Y)$ denotes the Y-periodic functions which belong to $V_{\rm loc}(\mathbb{R}^2)$;
- for any locally compact subset X of $\mathbb{R}^2$, $\mathcal{M}(X)$ denotes the space of the Radon measures defined on X;
- c denotes a constant which may vary form a line to another one.

2 Statement of the results

2.1 The general case

We consider a sequence of two-dimensional conduction problems in which the conductivity matrix-valued is either not uniformly bounded from above or (exclusively) not equicoercive. As a consequence, either the associated flux is not bounded in L^2 or the associated potential is not bounded in H^1. To take into account these two degenerate cases we extend the definition of the classical Murat-Tartar H-convergence (see [23]) by the following way:

Definition 2.1. Let α_n and β_n be two sequences of positive numbers such that $\alpha_n \leq \beta_n$, and let A_n be a sequence of matrix-valued functions in $M(\alpha_n, \beta_n; \Omega)$ (see (1.3)).

- The sequence A_n is said to $H(\mathcal{M}(\Omega)^2)$-converge to the matrix-valued function $A_* \in M(\alpha, \beta; \Omega)$, with $0 < \alpha \leq \beta$, if for any distribution f in $H^{-1}(\Omega)$, the solution u_n of the problem

$$\begin{cases} -\operatorname{div}(A_n \nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

satisfies the convergences

$$u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega) \qquad \text{and} \qquad A_n \nabla u_n \rightharpoonup A_* \nabla u \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2, \tag{2.2}$$

where u is the solution of the problem

$$\begin{cases} -\operatorname{div}(A_* \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We denote this convergence by $A_n \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} A_*$.

- The sequence A_n is said to $H(L^2(\Omega)^2)$-converge to the matrix-valued function $A_* \in M(\alpha, \beta; \Omega)$, with $0 < \alpha \leq \beta$, if for any function f in $L^2(\Omega)$, the solution u_n of (2.1) satisfies the convergences

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } L^2(\Omega) \\ u_n \longrightarrow u & \text{strongly in } L^2_{\text{loc}}(\Omega) \end{cases} \qquad \text{and} \qquad A_n \nabla u_n \rightharpoonup A_* \nabla u \quad \text{weakly in } L^2(\Omega)^2, \tag{2.4}$$

where u is the solution of (2.3). We denote this convergence by $A_n \overset{H(L^2(\Omega)^2)}{\rightharpoonup} A_*$.

The main result of the paper is the following:

Theorem 2.2. *Let Ω be a bounded open set of $\mathbb{R}^2$ such that $|\partial\Omega| = 0$. Let $\alpha > 0$, let β_n, $n \in \mathbb{N}$, be a sequence of real numbers such that $\beta_n \geq \alpha$, and let A_n be a sequence of matrix-valued functions (not necessarily symmetric) in $M(\alpha, \beta_n; \Omega)$.*
i) Assume that there exists a function $a \in L^\infty(\Omega)$ such that

$$\frac{\det A_n}{\det A_n^s} \, |A_n^s| \rightharpoonup a \quad \textit{weakly-}* \textit{ in } \mathcal{M}(\bar{\Omega}). \tag{2.5}$$

Then, there exists a subsequence of n, still denoted by n, and a matrix-valued function A_* in $M(\alpha, \beta; \Omega)$, with $\beta = 2 \, \|a\|_{L^\infty(\Omega)}$, such that

$$A_n \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} A_* \qquad \textit{and} \qquad A_n^T \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} A_*^T. \tag{2.6}$$

ii) In addition to the assumptions of i), assume that there exists a constant $C_0 > 0$ such that, for any $n \in \mathbb{N}$,

$$\frac{\det A_n}{\det A_n^s} \, A_n^s \leq C_0 \, A_n A_n^T, \quad \textit{a.e. in } \Omega. \tag{2.7}$$

Then, we have

$$\frac{A_n^T}{\det A_n} \overset{H(L^2(\Omega)^2)}{\rightharpoonup} \frac{A_*^T}{\det A_*}. \tag{2.8}$$

Remark 2.3. The part *i)* is a two-dimensional extension of the H-convergence for unbounded sequences of equicoercive matrix-valued functions. It was first proved in [8] under the following assumption: there exists a constant $\gamma > 0$ and $\bar{a} \in L^\infty(\Omega)$ such that $A_n = A_n^s + a_n J$ satisfies

$$|a_n| \le \gamma\, A_n^s \qquad \text{and} \qquad |A_n^s| \rightharpoonup \bar{a} \quad \text{weakly-}* \text{ in } \mathcal{M}(\bar{\Omega}). \tag{2.9}$$

Assumption (2.9) is more restrictive than (2.5) since

$$\frac{\det A_n}{\det A_n^s}\, |A_n^s| = \left(1 + \frac{a_n^2}{\det A_n^s}\right) |A_n^s| \le (1+\gamma^2)\, |A_n^s|$$

which converges to a bounded function in the weak-$*$ sense of the measures on $\bar{\Omega}$, hence convergence (2.5). The proof of (2.6) is quite similar to the one in [8] up to a few extra computations (see [20] for details).

On the contrary, the part *ii)* of Theorem 2.2 is a new result which extends the duality result obtained by Dykhne [11] for periodic and uniformly bounded conductivities to non-periodic and non-uniformly bounded ones. Condition (2.7) is a technical assumption we need in the non-symmetric case. Indeed, (2.7) clearly holds with $C_0 = \alpha^{-1}$, if $A_n \ge \alpha I$ is symmetric. It also holds if $A_n = \alpha_n I + a_n J$ (i.e. A_n^s is isotropic) with $\alpha_n \ge \alpha$, since

$$\frac{\det A_n}{\det A_n^s}\, A_n^s = \left(\frac{\alpha_n^2 + a_n^2}{\alpha_n}\right) I \le \left(\frac{\alpha_n^2 + a_n^2}{\alpha}\right) I = \alpha^{-1}\, A_n A_n^T.$$

Part *ii)* will be proved in Section 3.

Theorem 2.2 implies the following H-convergence result for uniformly bounded sequences of matrix-valued functions which are not equicoercive:

Corollary 2.4. *Let Ω be a bounded open set of $\mathbb{R}^2$ such that $|\partial\Omega| = 0$. Let $\beta > 0$ and let α_n be a sequence of real numbers such that $0 < \alpha_n \le \beta$. Let B_n be a sequence of matrix-valued functions in $M(\alpha_n, \beta; \Omega)$. Assume that there exist a function a in $L^\infty(\Omega)$ such that*

$$\left|(B_n^s)^{-1}\right| \rightharpoonup a \quad \text{weakly-}* \text{ in } \mathcal{M}(\bar{\Omega}), \tag{2.10}$$

and a constant $C_0 > 0$ such that, for any $n \in \mathbb{N}$,

$$B_n^T B_n \le C_0\, B_n^s, \quad \text{a.e. in } \Omega. \tag{2.11}$$

Then, there exists a subsequence of n, still denoted by n, and a matrix-valued function B_* in $M(\alpha, \beta; \Omega)$, with $\alpha = \left(2\,\|a\|_{L^\infty(\Omega)}\right)^{-1}$, such that

$$B_n \overset{H(L^2(\Omega)^2)}{\rightharpoonup} B_*. \tag{2.12}$$

Proof. The sequence A_n defined by

$$A_n := \frac{B_n^T}{\det B_n} = J^{-1} B_n^{-1} J,$$

satisfies the inequality $A_n \ge \beta^{-1} I$. Inequality (2.7) is a consequence of (2.11) since $B_n = J^{-1} A_n^{-1} J$ and

$$A_n A_n^T = J^{-1}(B_n^T B_n)^{-1} J \ge C_0^{-1}\, J^{-1}(B_n^s)^{-1} J = C_0^{-1}\, \frac{B_n^s}{\det B_n^s} = C_0^{-1}\, \frac{\det A_n}{\det A_n^s}\, A_n^s. \tag{2.13}$$

Moreover, convergence (2.5) is a consequence of (2.10) since

$$\left|(B_n^s)^{-1}\right| = \left|J^{-1}(B_n^s)^{-1} J\right| = \left|\frac{B_n^s}{\det B_n^s}\right| = \frac{\det A_n}{\det A_n^s}\, |A_n^s|. \tag{2.14}$$

Then, by the part $i)$ of Theorem 2.2, the sequence A_n (up to a subsequence) $H(\mathcal{M}(\Omega)^2)$-converges to some A_* in $M\left(\beta^{-1}, 2\|a\|_{L^\infty(\Omega)}; \Omega\right)$. Therefore, by the part $ii)$ of Theorem 2.2, B_n $H(L^2(\Omega)^2)$-converges to the matrix-valued function

$$B_* := \frac{A_*^T}{\det A_*} = J^{-1} A_*^{-1} J.$$

The matrix-valued function B_* clearly belongs to the set $M(\alpha, \beta; \Omega)$, with $\alpha := \left(2\|a\|_{L^\infty(\Omega)}\right)^{-1}$, which concludes the proof. □

2.2 The periodic case

In this section we consider the case of highly oscillating sequences of conductivity matrices. Let Ω be a bounded open subset of $\mathbb{R}^2$, and let $Y := (0,1)^2$ be the unit square of $\mathbb{R}^2$. Let $A_n^\sharp$ be a sequence of Y-periodic matrix-valued functions in $L^\infty_\#(\mathbb{R}^2)^{2\times 2}$, and let ε_n be a sequence of positive numbers which tends to 0. We define the highly oscillating sequence associated with $A_n^\sharp$ and ε_n by

$$A_n(x) := A_n^\sharp\left(\frac{x}{\varepsilon_n}\right), \quad \text{for a.e. } x \in \Omega. \tag{2.15}$$

For a fixed $n \in \mathbb{N}$, let A_n^* be the constant matrix defined by

$$A_n^*\lambda := \int_Y A_n^\sharp \nabla W_n^\lambda dy, \tag{2.16}$$

where W_n^λ, for $\lambda \in \mathbb{R}^2$, is the unique solution in $H^1_{\text{loc}}(\mathbb{R}^2)$ of the problem

$$\begin{cases} \operatorname{div}\left(A_n^\sharp \nabla W_n^\lambda\right) = 0 & \text{in } \mathbb{R}^2 \\ W_n^\lambda(y) - \lambda \cdot y & \text{is } Y\text{-periodic, with zero } Y\text{-average.} \end{cases} \tag{2.17}$$

Note that A_n^* is the H-limit of the oscillating sequence $A_n^\sharp(\frac{x}{\varepsilon})$ as ε tends to 0 (see e.g. the periodic homogenization in [4]). Under the periodicity assumption (2.15) we can improve Theorem 2.2. To this end, we need a more general definition of H-convergence than the one of Definition 2.1:

Definition 2.5. Let α_n and β_n be two sequences of positive numbers such that $\alpha_n \leq \beta_n$, and let A_n be a sequence of matrix-valued functions in $M(\alpha_n, \beta_n; \Omega)$.

- The sequence A_n is said to H_s-converge to the matrix-valued function $A_* \in M(\alpha, \beta; \Omega)$, with $0 < \alpha \leq \beta$, if for any function f in $L^2(\Omega)$, the solution u_n of problem (2.1) strongly converges in $L^2(\Omega)$ to the solution u of problem (2.3).

  We denote this convergence by $A_n \xrightarrow{H_s} A_*$.

- The sequence A_n is said to H_w-converge to the matrix-valued function $A_* \in M(\alpha, \beta; \Omega)$, with $0 < \alpha \leq \beta$, if for any function f in $L^2(\Omega)$, the solution u_n of problem (2.1) weakly converges in $L^2(\Omega)$ to the solution u of problem (2.3) and the flux $A_n \nabla u_n$ weakly converges to $A_* \nabla u$ in $L^2(\Omega)^2$.

  We denote this convergence by $A_n \xrightarrow{H_w} A_*$.

Remark 2.6. In the part $i)$ of Definition 2.5 we have the strong convergence of the potential but not the convergence of the flux. This corresponds to the case of an equicoercive sequence of conductivity matrices without control from above. In the part $ii)$ we have the weak convergence of both the potential and the flux. This corresponds to the case of a uniformly bounded sequence of conductivity matrices without control from below.

We have the following periodic homogenization result:

Theorem 2.7. *Let $\alpha > 0$ and let β_n be a sequence of real numbers such that $\beta_n \geq \alpha$. Let $A_n^\sharp$ be a sequence of Y-periodic matrix-valued functions (not necessarily symmetric) in $M(\alpha, \beta_n; \mathbb{R}^2)$, and let A_n be the highly oscillating sequence associated with $A_n^\sharp$ by (2.15).*
i) Assume that the sequence A_n^* defined by (2.16) converges to A_* in $\mathbb{R}^{2\times 2}$, and that the following limit holds

$$\varepsilon_n^2 \int_Y \frac{\det A_n^\sharp}{\det (A_n^\sharp)^s} \left|(A_n^\sharp)^s\right| dy \underset{n\to+\infty}{\longrightarrow} 0. \tag{2.18}$$

Then, we have

$$A_n \xrightarrow{H_s} A_*. \tag{2.19}$$

ii) In addition to the assumptions of i) assume that A_n and A_n^T satisfy inequality (2.7), and that the solution u_n of (2.1), with the matrix $A_n^T/\det A_n$, is bounded in $L^2(\Omega)$ for any right-hand side f in $L^2(\Omega)$. Then, we have

$$\frac{A_n^T}{\det A_n} \xrightarrow{H_w} \frac{A_*^T}{\det A_*}. \tag{2.20}$$

Remark 2.8. In the part *i)* of Theorem 2.7, taking into account the periodicity (2.15) convergence (2.5) is equivalent to the $L^1(Y)$-boundedness

$$\int_Y \frac{\det A_n^\sharp}{\det (A_n^\sharp)^s} \left|(A_n^\sharp)^s\right| dy \leq c,$$

which is clearly more restrictive than condition (2.18). The price to pay is that the sequence $A_n \nabla u_n$ is not necessarily bounded in $L^1(\Omega)^2$.

In the part *ii)* of Theorem 2.7 we have to assume the $L^2(\Omega)$-boundedness of any solution of (2.1) with conductivity matrix $A_n^T/\det A_n$, since condition (2.18) does not imply it. To this end, it is sufficient to assume the existence of a constant $C > 0$ such that, for any $n \in \mathbb{N}$,

$$\forall\, u \in H_0^1(\Omega), \qquad \int_\Omega u^2\, dx \leq C \int_\Omega \frac{A_n}{\det A_n} \nabla u \cdot \nabla u\, dx. \tag{2.21}$$

Example 2.9. Let E be a Y-periodic connected open set of $\mathbb{R}^2$, with a Lipschitz boundary, such that $|Y \cap E| > 0$. Consider a Y-periodic symmetric matrix-valued function $A_n^\sharp$ such that

$$\frac{A_n^\sharp}{\det A_n^\sharp} \geq I \quad \text{a.e. in } E \qquad \text{and} \qquad \frac{A_n^\sharp}{\det A_n^\sharp} \geq \varepsilon_n^2\, I \quad \text{a.e. in } \mathbb{R}^2 \setminus E,$$

or equivalently

$$A_n^\sharp \leq I \quad \text{a.e. in } E \qquad \text{and} \qquad A_n^\sharp \leq \varepsilon_n^{-2}\, I \quad \text{a.e. in } \mathbb{R}^2 \setminus E.$$

Then, the highly oscillating sequence A_n defined by (2.15) satisfies the Poincaré inequality (2.21) (see e.g. [2] for the derivation of a similar estimate). The proof of (2.21) is based on the extension property established in [1] (see [20] for more details).

3 Proof of the results

3.1 Proof of Theorem 2.2

Taking into account Remark 2.3 we focus on the part *ii)* of Theorem 2.2. Consider a sequence A_n in $M(\alpha, \beta_n; \Omega)$ which satisfies convergence (2.5) and $H(\mathcal{M}(\Omega)^2)$-converges to A_* in $M(\alpha, \beta; \Omega)$, with $0 < \alpha \leq \beta$, and set $B_n := J^{-1} A_n^{-1} J$. Let $f \in L^2(\Omega)$ and let v_n be the solution of the conduction problem (2.1) with conductivity matrix B_n. The proof of the $H(L^2(\Omega)^2)$-convergence (2.8) is divided into two steps. In the first step, we prove that the sequence v_n strongly converges in $L^2_{\text{loc}}(\Omega)$ to some $v \in H_0^1(\Omega)$, and that the flux $B_n \nabla v_n$ weakly converges to some ξ in $L^2(\Omega)$. The second step is devoted to the determination of the limit ξ in order to establish convergence (2.8).

First step : Convergences of the sequences v_n and $B_n\nabla v_n$.
Putting the function $v_n \in H^1_0(\Omega)$ as test function in the equation $-\operatorname{div}(B_n\nabla v_n) = f$, we obtain by the Sobolev embedding of $W^{1,1}(\Omega)$ into $L^2(\Omega)$ combined with the Poincaré inequality

$$\int_\Omega B_n\nabla v_n\cdot\nabla v_n\,dx = \int_\Omega f\,v_n\,dx \le \|f\|_{L^2(\Omega)}\,\|v_n\|_{L^2(\Omega)} \le c\int_\Omega |\nabla v_n|\,dx. \tag{3.1}$$

Moreover, by the Cauchy-Schwarz inequality combined with (2.14) we have

$$\begin{aligned}\int_\Omega |\nabla v_n|\,dx &\le \int_\Omega \left|(B_n^s)^{-\frac12}\right|\left|(B_n^s)^{\frac12}\nabla v_n\right|dx\\ &\le \left(\int_\Omega \left|(B_n^s)^{-1}\right|dx\right)^{\frac12}\left(\int_\Omega B_n^s\nabla v_n\cdot\nabla v_n\,dx\right)^{\frac12}\\ &= \left(\int_\Omega \frac{\det A_n}{\det A_n^s}\,|A_n^s|\,dx\right)^{\frac12}\left(\int_\Omega B_n\nabla v_n\cdot\nabla v_n\,dx\right)^{\frac12}.\end{aligned}$$

Then, we deduce from the previous inequalities and (2.5) that

$$\int_\Omega B_n\nabla v_n\cdot\nabla v_n\,dx \le c\left(\int_\Omega B_n\nabla v_n\cdot\nabla v_n\,dx\right)^{\frac12}. \tag{3.2}$$

Therefore, the sequences $B_n\nabla v_n\cdot\nabla v_n$ and $|\nabla v_n|$ are bounded in $L^1(\Omega)$, hence v_n is bounded in $L^2(\Omega)$ by (3.1). On the other hand, similarly to (2.13) inequality (2.7) implies that $B_n^T B_n \le C_0\,B_n^s$ and

$$|B_n\nabla v_n|^2 = (B_n^T B_n)\nabla v_n\cdot\nabla v_n \le C_0\,B_n^s\nabla v_n\cdot\nabla v_n = C_0\,B_n\nabla v_n\cdot\nabla v_n,$$

hence the sequence $B_n\nabla v_n$ is also bounded in $L^2(\Omega)$. Therefore, up to a subsequence v_n weakly converges to v in $L^2(\Omega)$ and $B_n\nabla v_n$ weakly converges to ξ in $L^2(\Omega)^2$.

The strong convergence of v_n in $L^2_{\rm loc}(\Omega)$ is a consequence of the following result which is proved in [8] (see the steps $3,4$ of the proof of Theorem 2.1 in [8], as well as the first step of Theorem 2.7 *i*), which uses similar arguments adapted to condition (2.18)):

Lemma 3.1. *Let S_n be a sequence of symmetric matrix-valued functions in $L^\infty(\Omega)^{2\times2}$ such that there exist $\alpha>0$ and $a\in L^\infty(\Omega)$ satisfying*

$$S_n \ge \alpha\,I \qquad \text{and} \qquad |S_n| \rightharpoonup a \quad \text{weakly-}*\text{ in } \mathcal{M}(\Omega). \tag{3.3}$$

Let v_n be a sequence in $H^1(\Omega)$ satisfying

$$v_n \rightharpoonup v \quad \text{weakly in } L^2(\Omega) \qquad \text{and} \qquad \int_\Omega S_n^{-1}\nabla v_n\cdot\nabla v_n\,dx \le c. \tag{3.4}$$

Then, the sequence v_n strongly converges to v in $L^2_{\rm loc}(\Omega)$.

Set $S_n := (B_n^s)^{-1}$. Since $A_n \ge \alpha\,I$, we have $|B_n^s| \le |B_n| = |A_n^{-1}| \le \alpha^{-1}$, hence $B_n^s \le \alpha^{-1}I$ and $S_n \ge \alpha\,I$. Moreover, by (2.5) and (2.14) S_n satisfies the weak convergence of (3.3), and by (3.2) v_n satisfies (3.4). Lemma 3.1 thus implies that v_n strongly converges to v in $L^2_{\rm loc}(\Omega)$.

It remains to prove that v belongs to $H^1_0(\Omega)$. Let $\Phi\in C^1(\bar\Omega)^2$. Using successively the Cauchy-Schwarz inequality and (3.2) we have

$$\begin{aligned}\left|\int_\Omega v_n\operatorname{div}\Phi\,dx\right| &= \left|\int_\Omega \Phi\cdot\nabla v_n\,dx\right| = \left|\int_\Omega (B_n^s)^{-\frac12}\Phi\cdot(B_n^s)^{\frac12}\nabla v_n\,dx\right|\\ &\le \left(\int_\Omega \left|(B_n^s)^{-1}\right||\Phi|^2\,dx\right)^{\frac12}\left(\int_\Omega B_n\nabla v_n\cdot\nabla v_n\,dx\right)^{\frac12}\\ &\le c\left(\int_\Omega \left|(B_n^s)^{-1}\right||\Phi|^2\,dx\right)^{\frac12}.\end{aligned}$$

Therefore, passing to the limit in the previous inequality thanks to the weak convergence of v_n, to equality (2.14) and to convergence (2.5), we get

$$\left| \int_\Omega v \operatorname{div} \Phi \, dx \right| \leq c \, \|a\|^{\frac{1}{2}}_{L^\infty(\Omega)} \, \|\Phi\|_{L^2(\Omega)^2}, \quad \text{for any } \Phi \in C^1(\bar{\Omega})^2,$$

which implies that v belongs to $H^1_0(\Omega)$.

Second step : Determination of the limit ξ of $B_n \nabla v_n$.
Let $\lambda \in \mathbb{R}^2$, $\theta \in C^1_c(\Omega)$, and let w^λ_n be the solution of the problem

$$\begin{cases} \operatorname{div}\left(A^T_n \nabla w^\lambda_n\right) = \operatorname{div}\left(A^T_* \nabla(\theta\, \lambda \cdot x)\right) & \text{in } \Omega \\ w^\lambda_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

By (2.6) and by virtue of Definition 2.1 we have the following convergences

$$w^\lambda_n \rightharpoonup \theta\, \lambda \cdot x \quad \text{weakly in } H^1_0(\Omega) \qquad \text{and} \qquad A^T_n \nabla w^\lambda_n \rightharpoonup A^T_* \nabla(\theta\, \lambda \cdot x) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2. \tag{3.6}$$

Now, we will pass to the limit in the product $B_n \nabla v_n \cdot J A^T_n \nabla w^\lambda_n$ by two different ways, which will give the desired limit ξ.

On the one hand, since $B_n = J^{-1} A^{-1}_n J$ and $J^2 = -I$, we have

$$B_n \nabla v_n \cdot J A^T_n \nabla w^\lambda_n = -A^{-1}_n J \nabla v_n \cdot A^T_n \nabla w^\lambda_n = -J \nabla v_n \cdot \nabla w^\lambda_n = \nabla v_n \cdot J \nabla w^\lambda_n.$$

Moreover, since $J\nabla w^\lambda_n$ is divergence free, we have $\nabla v_n \cdot J \nabla w^\lambda_n = \operatorname{div}\left(v_n\, J \nabla w^\lambda_n\right)$. Then, since v_n strongly converges to v in $L^2_{\text{loc}}(\Omega)$ and ∇w^λ_n weakly converges to $\nabla(\theta\, \lambda \cdot x)$ in $L^2(\Omega)^2$ by (3.6), the sequence $v_n\, J \nabla w^\lambda_n$ converges to $v\, J \nabla(\theta\, \lambda \cdot x)$ in $L^1_{\text{loc}}(\Omega)$. Therefore, we obtain the first convergence

$$B_n \nabla v_n \cdot J A^T_n \nabla w^\lambda_n \rightharpoonup \operatorname{div}\left(v\, J \nabla(\theta\, \lambda \cdot x)\right) = \nabla v \cdot J \nabla(\theta\, \lambda \cdot x) \quad \text{in } \mathcal{D}'(\Omega). \tag{3.7}$$

On the other hand, consider a regular simply connected open subset ω of Ω. Since by definition (3.5) $A^T_n \nabla w^\lambda_n - A^T_* \nabla(\theta\, \lambda \cdot x)$ is a divergence free function in $L^2(\omega)^2$, there exists a stream function (see e.g. [14]) $\tilde{w}^\lambda_n$ in $H^1(\omega)$ uniquely defined by

$$\int_\omega \tilde{w}^\lambda_n \, dx = 0 \quad \text{and} \quad A^T_n \nabla w^\lambda_n - A^T_* \nabla(\theta\, \lambda \cdot x) = J \nabla \tilde{w}^\lambda_n. \tag{3.8}$$

Since $A^T_n \nabla w^\lambda_n$ is bounded in $L^1(\Omega)^2$ by (3.6) and $\tilde{w}^\lambda_n$ has a zero ω-average, the Sobolev imbedding of $W^{1,1}(\omega)$ into $L^2(\omega)$ combined with the Poincaré-Wirtinger inequality in ω implies that $\tilde{w}^\lambda_n$ is bounded in $L^2(\omega)$ and thus converges, up to a subsequence, to a function $\tilde{w}^\lambda$ in $L^2(\omega)$. Moreover, by the Cauchy-Schwarz inequality and (3.5) we have, with $B_n = J^{-1} A^{-1}_n J$,

$$\begin{aligned}
&\int_\omega B^s_n \nabla \tilde{w}^\lambda_n \cdot \nabla \tilde{w}^\lambda_n \, dx = \int_\omega \left(A^{-1}_n\right)^s J \nabla \tilde{w}^\lambda_n \cdot J \nabla \tilde{w}^\lambda_n \, dx \\
&= \int_\omega \left(A^{-1}_n\right)^s \left[A^T_n \nabla w^\lambda_n - A^T_* \nabla(\theta\, \lambda \cdot x)\right] \cdot \left[A^T_n \nabla w^\lambda_n - A^T_* \nabla(\theta\, \lambda \cdot x)\right] dx \\
&\leq 2 \int_\omega \left(A^{-1}_n\right)^s A^T_n \nabla w^\lambda_n \cdot A^T_n \nabla w^\lambda_n + \left(A^{-1}_n\right)^s A^T_* \nabla(\theta\, \lambda \cdot x) \cdot A^T_* \nabla(\theta\, \lambda \cdot x) \, dx \\
&\leq 2 \int_\Omega A^T_n \nabla w^\lambda_n \cdot \nabla w^\lambda_n + A^{-1}_n A^T_* \nabla(\theta\, \lambda \cdot x) \cdot A^T_* \nabla(\theta\, \lambda \cdot x) \, dx \\
&= 2 \int_\Omega A^T_* \nabla(\theta\, \lambda \cdot x) \cdot \nabla w^\lambda_n + A^{-1}_n A^T_* \nabla(\theta\, \lambda \cdot x) \cdot A^T_* \nabla(\theta\, \lambda \cdot x) \, dx.
\end{aligned}$$

The last term is bounded by (3.6) and by the inequality $|A^{-1}_n| \leq \alpha^{-1}$. Therefore, the sequences $v_n := \tilde{w}^\lambda_n$ and $S_n = (B^s_n)^{-1}$ of the first step satisfy the assumptions (3.3) and (3.4) of Lemma 3.1

in ω, hence $\tilde{w}_n^\lambda$ strongly converges to $\tilde{w}^\lambda$ in $L^2_{\rm loc}(\omega)$. Moreover, the second convergence of (3.6) and definition (3.8) imply that $\tilde{w}^\lambda$ has a zero ω-average and $\nabla\tilde{w}^\lambda = 0$ in $\mathcal{D}'(\omega)$, hence $\tilde{w}^\lambda = 0$ by the connectedness of ω. Therefore, by the uniqueness of the limit we get for the whole sequence

$$\tilde{w}_n^\lambda \longrightarrow 0 \quad \text{strongly in } L^2_{\rm loc}(\omega). \tag{3.9}$$

By (3.8) we have

$$B_n\nabla v_n\cdot JA_n^T\nabla w_n^\lambda = B_n\nabla v_n\cdot JA_*^T\nabla(\theta\,\lambda\cdot x) - B_n\nabla v_n\cdot\nabla\tilde{w}_n^\lambda.$$

Clearly, the sequence $B_n\nabla v_n\cdot JA_*^T\nabla(\theta\,\lambda\cdot x)$ weakly converges to $\xi\cdot JA_*^T\nabla(\theta\,\lambda\cdot x)$ in $L^2(\omega)^2$. Moreover, the strong convergence (3.9) implies that

$$B_n\nabla v_n\cdot\nabla\tilde{w}_n^\lambda = {\rm div}\left(\tilde{w}_n^\lambda\,B_n\nabla v_n\right) + \tilde{w}_n^\lambda\,f \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\omega).$$

Therefore, we obtain

$$B_n\nabla v_n\cdot JA_n^T\nabla w_n^\lambda \rightharpoonup \xi\cdot JA_*^T\nabla(\theta\,\lambda\cdot x) \quad \text{in } \mathcal{D}'(\omega).$$

This combined with (3.7) yields

$$\nabla v\cdot J\nabla(\theta\,\lambda\cdot x) = \xi\cdot JA_*^T\nabla(\theta\,\lambda\cdot x) \quad \text{a.e. in } \omega.$$

Now, choose $\theta\in C^1_c(\Omega)$ such that $\theta = 1$ in ω in the former equality. Therefore, due to the arbitrariness of λ and ω we get the equality $J\nabla v = A_*J\xi$ a.e. in Ω, hence $\xi = J^{-1}A_*^{-1}J\nabla v = B_*\nabla v$ a.e. in Ω, which concludes the proof.

3.2 Proof of Theorem 2.7

Proof of the part *i)* **of Theorem 2.7.** The proof is similar to the one of the compactness result in [5]. But there are extra difficulties since the conductivity matrices are not symmetric and the fluxes are not necessarily bounded in $L^1(\Omega)$, due to the condition (2.18). We will give the main steps of the proof pointing out these difficulties.

Let u_n be the solution of the conduction problem (2.1), where A_n is the highly ocillating sequence (2.15). Let $\lambda\in\mathbb{R}^2$, and let V_n^λ be the unique solution of problem (2.17) with the matrix-valued function $(A_n^\sharp)^T$. Note that the matrix A_n^* defined by (2.16) and V_n^λ satisfy the relation

$$(A_n^*)^T\lambda = \int_Y (A_n^\sharp)^T\nabla V_n^\lambda\,dy \quad\text{and}\quad (A_n^*)^T\lambda\cdot\lambda = \int_Y (A_n^\sharp)^T\nabla V_n^\lambda\cdot\nabla V_n^\lambda\,dy \le c\,|\lambda|^2. \tag{3.10}$$

Set $v_n^\lambda(x) := \varepsilon_n V_n^\lambda(\frac{x}{\varepsilon_n})$ and $z_n^\lambda(x) := v_n^\lambda(x) - \lambda\cdot x$. Note that the second estimate of (3.10) and the α-coerciveness of $A_n^\sharp$ imply that the sequence $(V_n^\lambda - \lambda\cdot y)$ is bounded in $H^1_\#(Y)$, hence

$$z_n^\lambda \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega). \tag{3.11}$$

To prove the H_s-convergence (2.19) it is enough to prove that

$$A_n\nabla u_n \rightharpoonup A_*\nabla u \quad \text{in } \mathcal{D}'(\Omega),$$

where A_* is the limit of A_n^* in $\mathbb{R}^{2\times2}$, and u is the weak limit of u_n in $H^1_0(\Omega)$. To this end, we proceed in two steps. In the first step, we prove the convergence

$$A_n\nabla u_n\cdot\nabla v_n^\lambda - A_n\nabla u_n\cdot\lambda \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{3.12}$$

and in the second one, the convergence

$$A_n\nabla u_n\cdot\nabla v_n^\lambda - A_*\nabla u\cdot\lambda \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.13}$$

First step : Proof of (3.12).
Let ω be a regular simply connected subset of Ω, let $v \in H_0^1(\Omega)$ be the solution of $-\Delta v = f$, and consider the stream function $\tilde{u}_n \in W^{1,1}(\omega)$ defined by

$$\int_\omega \tilde{u}_n \, dx = 0 \qquad \text{and} \qquad A_n \nabla u_n - \nabla v = J \nabla \tilde{u}_n \quad \text{a.e. in } \omega. \tag{3.14}$$

Set $\tilde{A}_n := J^{-1} \left(A_n^{-1} \right)^s J$ and $\tilde{A}_n^\sharp := J^{-1} \left[(A_n^\sharp)^{-1} \right]^s J$. Using successively the Poincaré-Wirtinger inequality in ω, the Cauchy-Schwarz inequality, equality (2.14), estimate (2.18) and $|A_n^{-1}| \le \alpha^{-1}$, we have

$$\begin{aligned} \int_\omega |\tilde{u}_n| \, dx &\le c \int_\omega |\nabla \tilde{u}_n| \, dx \\ &\le c \left(\int_\omega |\tilde{A}_n^{-1}| \, dx \right)^{\frac{1}{2}} \left(\int_\omega \tilde{A}_n \nabla \tilde{u}_n \cdot \nabla \tilde{u}_n \, dx \right)^{\frac{1}{2}} \\ &\le c \left(\int_Y |(\tilde{A}_n^\sharp)^{-1}| \, dy \right)^{\frac{1}{2}} \left(\int_\omega \tilde{A}_n \nabla \tilde{u}_n \cdot \nabla \tilde{u}_n \, dx \right)^{\frac{1}{2}} \\ &\le c \left(\int_Y \frac{\det A_n^\sharp}{\det (A_n^\sharp)^s} \left| (A_n^\sharp)^s \right| dy \right)^{\frac{1}{2}} \left(\int_\omega A_n \nabla u_n \cdot \nabla u_n + A_n^{-1} \nabla v \cdot \nabla v \, dx \right)^{\frac{1}{2}} \\ &= o\left(\varepsilon_n^{-1} \right). \end{aligned} \tag{3.15}$$

To get (3.12) we need to prove that the sequence $A_n \nabla u_n \cdot \nabla z_n^\lambda$ converges to zero in $\mathcal{D}'(\Omega)$. To this end consider $\varphi \in {}_c^\infty(\Omega)$. Integrating by parts we deduce from (3.14) and (3.11) the equality

$$\int_\omega A_n \nabla u_n \cdot \nabla z_n^\lambda \, \varphi \, dx = \int_\omega \nabla v \cdot \nabla z_n^\lambda \, \varphi \, dx + \int_\omega \tilde{u}_n \, J \nabla z_n^\lambda \cdot \nabla \varphi \, dx = \int_\omega \tilde{u}_n \, J \nabla z_n^\lambda \cdot \nabla \varphi \, dx + o(1). \tag{3.16}$$

Let $Q_n \subset \omega$ be a covering of $\operatorname{supp} \varphi$ by the squares $\varepsilon_n(k+Y)$, $k \in K_n \subset \mathbb{Z}^2$, and let $\bar{u}_n$ be the piecewise constant function defined by

$$\bar{u}_n := \sum_{k \in K_n} \left(\fint_{\varepsilon_n(k+Y)} \tilde{u}_n \right) 1_{\varepsilon_n(k+Y)}. \tag{3.17}$$

Following the procedure of [5], let us prove that $\bar{u}_n - \tilde{u}_n$ strongly converges to 0 on $\operatorname{supp} \varphi$. By the Sobolev imbedding of $W^{1,1}$ in L^2 in each square $\varepsilon_n(k+Y)$, $k \in K_n$, (note that the following imbedding constant C is independent of the squares) combined with the Poincaré-Wirtinger inequality, and by the Cauchy-Schwarz inequality we have

$$\begin{aligned} \int_{\varepsilon_n(k+Y)} (\bar{u}_n - \tilde{u}_n)^2 \, dx &\le C \left(\int_{\varepsilon_n(k+Y)} |\nabla \tilde{u}_n| \, dx \right)^2 \\ &\le C \int_{\varepsilon_n(k+Y)} |\tilde{A}_n^{-1}| \, dx \int_{\varepsilon_n(k+Y)} \tilde{A}_n \nabla \tilde{u}_n \cdot \nabla \tilde{u}_n \, dx. \end{aligned} \tag{3.18}$$

Then, summing over $k \in K_n$ we get similarly to (3.15)

$$\int_{Q_n} (\bar{u}_n - \tilde{u}_n)^2 \, dx \le c \, \varepsilon_n^2 \int_Y \frac{\det A_n^\sharp}{\det (A_n^\sharp)^s} \left| (A_n^\sharp)^s \right| dy \int_\omega \left(A_n \nabla u_n \cdot \nabla u_n + A_n^{-1} \nabla v \cdot \nabla v \right) dx, \tag{3.19}$$

which tends to 0 by (2.18). Therefore, we can replace $\tilde{u}_n$ by $\bar{u}_n$ in (3.16). Now, consider the approximation of $\nabla \varphi$ by a function $\bar{\Phi}_n$ constant in each square $\varepsilon_n(k+Y)$ and such that $|\nabla \varphi - \bar{\Phi}_n| \le c \, \varepsilon_n$.

Then, since $\nabla V_n - \lambda$ has a zero Y-average, the last term of (3.16) reads as

$$\begin{aligned}\int_\omega \tilde{u}_n\, J\nabla z_n^\lambda \cdot \nabla\varphi\, dx &= \int_\omega \bar{u}_n\, J\nabla z_n^\lambda \cdot \bar{\Phi}_n\, dx + \int_\omega \bar{u}_n\, J\nabla z_n^\lambda \cdot (\nabla\varphi - \bar{\Phi}_n)\, dx + o(1) \\ &= \int_\omega \bar{u}_n\, J\nabla z_n^\lambda \cdot (\nabla\varphi - \bar{\Phi}_n)\, dx + o(1).\end{aligned}$$

Using $|\nabla\varphi - \bar{\Phi}_n| \le c\,\varepsilon_n$, estimate (3.15) and the one of (3.10), we also have

$$\begin{aligned}\left|\int_\omega \bar{u}_n\, J\nabla z_n^\lambda \cdot (\bar{\Phi}_n - \nabla\varphi)\, dx\right| &\le c\,\varepsilon_n \int_{Q_n} |\bar{u}_n|\, |\nabla z_n^\lambda|\, dx \\ &= c\,\varepsilon_n \int_Y |\nabla V_n^\lambda - \lambda|\, dy \int_{Q_n} |\bar{u}_n|\ dx \\ &\le c\,\varepsilon_n \int_\omega |\tilde{u}_n|\ dx = o(1).\end{aligned}$$

The two previous estimates combined with (3.16) conclude the first step.

Second step : Proof of (3.13).
Following the first step and taking into account that $(A_n^\sharp)^T\nabla V_n^\lambda$ is a periodic divergence free function, we may define the periodic stream function $\tilde{V}_n^\lambda \in H^1_\#(Y)$ by

$$\int_Y \tilde{V}_n^\lambda\, dy = 0 \quad \text{and} \quad (A_n^\sharp)^T\nabla V_n^\lambda = \int_Y (A_n^\sharp)^T\nabla V_n^\lambda\, dy + J\nabla\tilde{V}_n^\lambda = (A_n^*)^T\lambda + J\nabla\tilde{V}_n^\lambda, \tag{3.20}$$

where the second equality is a consequence of (3.10). Proceeding similarly to (3.18) and (3.19), we have by the equality $\tilde{A}_n^\sharp = J^{-1}\big[(A_n^\sharp)^{-1}\big]^s J$ and estimates (2.18), (3.10),

$$\int_Y (\tilde{V}_n^\lambda)^2\, dy \le \int_Y \frac{\det A_n^\sharp}{\det(A_n^\sharp)^s}\left|(A_n^\sharp)^s\right| dy \int_Y \Big[(A_n^\sharp)^T\nabla V_n^\lambda\cdot\nabla V_n^\lambda + (A_n^\sharp)^{-1}(A_n^*)^T\lambda\cdot(A_n^*)^T\lambda\Big]\, dy = o\left(\varepsilon_n^{-2}\right),$$

hence the sequence $\tilde{v}_n^\lambda(x) := \varepsilon_n \tilde{V}_n^\lambda(\frac{x}{\varepsilon_n})$ strongly converges to 0 in $L^2(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$. Therefore, using the second equality of (3.20) and integrating by parts we get

$$\begin{aligned}\int_\Omega A_n\nabla u_n \cdot \nabla v_n^\lambda\,\varphi\, dx = \int_\Omega \nabla u_n \cdot A_n^T\nabla v_n^\lambda\,\varphi\, dx &= \int_\Omega \nabla u_n \cdot (A_n^*)^T\lambda\,\varphi\, dx + \int_\Omega \nabla u_n \cdot J\nabla\tilde{v}_n^\lambda\,\varphi\, dx \\ &= \int_\Omega A_n^*\nabla u_n \cdot \lambda\,\varphi\, dx + \int_\Omega \tilde{v}_n^\lambda\, J\nabla u_n \cdot \nabla\varphi\, dx \\ &= \int_\Omega A_*\nabla u \cdot \lambda\,\varphi\, dx + o(1),\end{aligned}$$

which yields (3.13).

Proof of the part *ii)* **of Theorem 2.7.** Set $B_n := J^{-1}A_n^{-1}J$ and $B_n^\sharp := J^{-1}(A_n^\sharp)^{-1}J$. Let B_n^* be the constant matrix defined by formula (2.16) with the matrix-valued function $B_n^\sharp$. By the classical duality formula due to Dykhne [11] (see also [13]) we have $B_n^* = J^{-1}(A_n^*)^{-1}J$, where A_n^* is given by (2.16). Therefore, the sequence B_n^* converges to $B_* := J^{-1}(A_*)^{-1}J$, where A_* is the limit of A_n^*.

On the other hand, for any periodic function $V \in H^1_\#(Y)$ with Y-average $\bar{V}$, the Sobolev imbedding of $W^{1,1}_\#(Y)$ into $L^2_\#(Y)$ combined with the Poincaré-Wirtinger inequality in Y, the Cauchy-Schwarz inequality and equality (2.14) with $B_n^\sharp$, imply that

$$\begin{aligned}\int_Y (V-\bar V)^2\,dy \le c\left(\int_Y |\nabla V|\,dy\right)^2 &\le c\left(\int_Y \left|\left[(B_n^\sharp)^s\right]^{-\frac12}\right|\left|\left[(B_n^\sharp)^s\right]^{\frac12}\nabla V\right|dy\right)^2\\ &\le c\left(\int_Y \left|\left[(B_n^\sharp)^s\right]^{-1}\right|dy\right)\int_Y (B_n^\sharp)^s\nabla V\cdot\nabla V\,dy\\ &= c\left(\int_Y \frac{\det A_n^\sharp}{\det(A_n^\sharp)^s}\left|(A_n^\sharp)^s\right|dy\right)\int_Y B_n^\sharp\nabla V\cdot\nabla V\,dy.\end{aligned}$$

This, combined with (2.18), yields the following estimate of the weighted Poincaré-Wirtinger inequality

$$\sup_{V\in H^1_\#(Y),\,V\neq\bar V}\left[\frac{\displaystyle\int_Y (V-\bar V)^2\,dy}{\displaystyle\int_Y B_n^\sharp\nabla V\cdot\nabla V\,dy}\right]\le C_n \qquad\text{with}\qquad \lim_{n\to+\infty}\varepsilon_n^2\,C_n=0. \tag{3.21}$$

In the symmetric case $B_n = B_n^s$, the first author proved in [7] that, under the $L^2(\Omega)$-boundedness of any solution v_n of $-\operatorname{div}(B_n\nabla v_n) = f \in L^2(\Omega)$, estimate (3.21) is a sufficient condition to obtain the H_w-convergence of B_n to B_*. This compactness result can be easily extended (see [20] for details) to the non-symmetric case assuming that A_n and A_n^T satisfy condition (2.7), or equivalently B_n and B_n^T satisfy (2.11). Therefore, the H_w-convergence (2.20) holds true since

$$B_n = \frac{A_n^T}{\det A_n} \quad\text{and}\quad B_* = \frac{A_*^T}{\det A_*},$$

which concludes the proof.

Acknowledgement. The authors wish to thank F. Murat for a remark which motivated this work.

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